Final Exam

Statistical Mechanics and Thermodynamics

SoSe 2018

3rd of July 2018

JGU Mainz - Institut für Physik

Each problem is worth 25 points.

You must complete 4 problems. You must choose one problem between Problem 1 and Problem 2. You must complete Problem 3 and 4. Then choose one problem from Problems 5, 6 and 7. Also, write your name on EACH of the papers that you hand in and number the pages.

Chose one problem between Problem 1 and Problem 2

Problem 1

1. For a general (non-ideal) gas, derive the formula

$$dE = C_V dT + \left[T \left(\frac{\partial P}{\partial T} \right)_V - P \right] dV.$$

   C_V is the constant-volume heat capacity, assumed here to be a constant. You may want to use the Maxwell relation,

$$\left(\frac{\partial S}{\partial V} \right)_T = \left(\frac{\partial P}{\partial T} \right)_V .$$

2. For the van-der-Waals equation of state,

$$\left(P + \frac{N^2 a}{V^2} \right) (V - b) = k_B N T,$$

   derive a specific form for dE given above. Here N denotes the number of particles. C_V is again a constant (valid, presumably, over a limited temperature range).

3. Find, for the van-der-Waals gas, the constant-pressure heat capacity $C_P(T, V)$, using your result from the previous part. Note that C_P is not a constant, but show that it reduces to the expected form for the ideal gas case. You can use the relation $C_P - C_V = T\,(\partial V/\partial T)_P\,(\partial P/\partial T)_V$.

Problem 2

A gas is described by the following equations of state

$$P = \frac{E}{3V}, \quad E = bVT^4$$

where b is a constant and P, E, V, and T are the thermodynamic pressure, internal energy, volume and temperature.

1. Determine the entropy S as a function of E and V.
2. Determine the functional relationship between P and V along an adiabatic path.
3. Determine the functional relationship between P and V along an isothermal path.
4. Determine the work done by the system as it expands from an initial volume V_0 to a final volume $2V_0$ along an isothermal path with temperature T_0.
5. Determine the heat added to the system as it expands from an initial volume V_0 to a final volume $2V_0$ along an isothermal path with temperature T_0.

Problem 3

For an ideal relativistic gas of noninteracting spin-1/2 quantum particles at $T = 0$ calculate the chemical potential μ, pressure P, and the internal energy E as a function of particle density $N/V = n$. The energy of the particles is described as $\varepsilon = c|\mathbf{p}|$, where $\mathbf{p}$ is the momentum.

Problem 4

Consider a gas of non-interacting Bose particles with spin $S = 0$ and mass m. In the ultrarelativistic limit, one can approximate the dispersion relation by $\varepsilon = c|\mathbf{p}|$, where $\mathbf{p}$ is the momentum..

1. Write down a general integral expression for the statistical average of the total number of particles not in the zero-energy ground state.

2. Determine the Bose-Einstein condensation temperature T_0 of the gas as a function of the gas density $\rho = N/V$.

3. Determine the fraction N_0/N of the particles in the zero-energy ground state as a function of temperature T and density ρ.

You may find the following formula useful:

$$\int_0^\infty \frac{z^{x-1}}{e^z - 1} dz = \Gamma(x)\xi(x)$$

x	3/ 2	5/ 2	3	5
Γ	$\sqrt{\pi}$ / 2	$3\sqrt{\pi}$ / 4	2	24
ξ	2.612	1.341	1.202	1.037

Chose one between Problem 5, Problem 6, and Problem 7

Problem 5

A hydrogen atom in equilibrium with a radiation field at temperature T can be in its ground state orbital level (the "$1-s$" level, which is two-fold spin degenerate with energy ϵ_0), or it can be in its first excited state energy level (the "$2-p$" level, which is six-fold degenerate with energy ϵ_1). For the purpose of this problem we shall assume that this atom does not have any other excited states (i.e., no $2s$ level and no levels with the principal quantum number $n > 2$).

1. (a) What is the probability that the atom will be in an "orbital s-state"?

   (b) What is the probability that the atom will be in an "orbital p-state"?

2. If the temperature is such that $k_B T = \epsilon_1 - \epsilon_0$, then show and state which of the two orbital levels is occupied more.

3. Derive an expression for the mean energy of the atom at temperature T and obtain the limiting value of this mean energy as $T \to \infty$.

4. Calculate the free energy and derive an expression for the entropy of the atom at temperature T and also, from the definition of entropy, state what should be the values of the entropy for this atom in the limits of $T \to 0$ and $T \to \infty$. (If you do not know the answer for the last question, you may obtain the limits from your general expression of entropy.)

Problem 6

Consider an extremely relativistic gas consisting of noninteracting N identical monoatomic molecules with energy momentum relationship $\varepsilon = c|\mathbf{p}|$, where c is the speed of light, and $\mathbf{p}$ is the momentum. The gas is confined to a volume V and is in thermal equilibrium at temperature T .

1. Calculate the partition function Z_N for the gas.
2. Calculate the Helmholtz free energy $F(T, V)$.
3. Calculate the internal energy E, the entropy S, and the heat capacity C_V of the gas.
4. Derive an equation of state of the gas.

Problem 7

Consider a system of N independent harmonic oscillators with the same frequency ω. The system is in equilibruim at a temperature T.

1. Show that the partition function of the system is

$$Z_N = \left[2\sinh\left(\frac{\hbar\omega}{2k_BT}\right)\right]^{-N}$$

2. Using this result, obtain the internal energy E of the system as a function of T and N.
3. Show that the heat capacity is

$$C = Nk_B \frac{e^{\hbar\omega/k_BT}}{\left(e^{\hbar\omega/k_BT} - 1\right)^2}\left(\frac{\hbar\omega}{k_BT}\right)^2$$

   and it approaches to 0 as $T \to 0$.

   Some trigonometric formula could be useful:

$$\text{cotanh}x \equiv \frac{\cosh x}{\sinh x}, \quad \frac{d}{dx}\sinh x = \cosh x, \quad \frac{d}{dx}\cosh x = \sinh x, \quad \cosh^2 x - \sinh^2 x = 1.$$

4. Determine its Helmholtz free energy F.
5. Determine its entropy S.