**Exercise Problems - Assignment 8**

# Statistical Mechanics and Thermodynamics

WS 2018

Due: 7th Jan 2019

JGU Mainz - Institut für Physik

## Choose 4 of the 6 problems below. Each problem is worth 12 points.

### Problem 1

1. Two containers of equal volume are filled with an equal number of moles of two different ideal monoatomic classical gasses. The containers are in thermal contact. The molecules of one of the gasses obey classical (Maxwellian, Boltzmann) statistics, while the molecules of the other gas obey Fermi-Dirac statistics. Which container has greater
   (a) pressure
   (b) internal energy
   (c) heat capacity at constant volume
   (d) entropy
2. Answer the same questions as for previous part, but comparing classical statistics and Bose-Einstein statistics. Do any of your answers depend on the temperature?

### Problem 2

1. Find the Fermi energy at $T = 0$, $\varepsilon_F$, of a gas of $N$ noninteracting spin one-half particles constrained to move in two dimensions within an area $A$.
2. The analogous of the pressure in two dimension is given by $-\partial\epsilon_n/\partial A$. Show that the 2-d pressure at $T = 0$ is given by $N\varepsilon_F/2A$.

### Problem 3

The relation between the frequency $\nu$ and the wavelength $\lambda$ for surface tension waves on the surface of a liquid of density $\rho$ and surface tension $\sigma$ is

$$\nu^2 = \frac{2\pi\sigma}{\rho\lambda^3}$$

Use a method analogous to the Debye theory of specific heats to find a formula, analogous to the Debye $T^3$ law, for the temperature dependence of the surface energy E of a liquid at low temperatures. The surface tension of liquid helium at $0K$ is $0.352 \times 10^{-3} N/m$ and its density is $0.145 g \cdot \text{cm}^{-3}$. From these data estimate the temperature range over which your formula for $E(T)$ is valid for liquid helium, assuming that each helium atom in the surface of the liquid possesses one degree of freedom. You may assume

$$\int_0^\infty \frac{x^{4/3}}{e^x - 1} dx = 1.68.$$

Bolzmann's constant $k_B = 1.38 \times 10^{-16}$ erg $\cdot K^{-1}$
Avogadro's number $N = 6.02 \times 10^{23}$ mole $^{-1}$
Planck's constant $\hbar = 1.05 \times 10^{-27}$ erg $\cdot$ sec

### Problem 4

The ground state density of a free-electron Fermi gas is conveniently parametrised by specifying the volume per conduction electron according to

$$\frac{4}{3}\pi r_s^3 \equiv \frac{V}{N}$$

1. Find the Fermi wavevector, $k_F$ in terms of the electron density parameter rs.

2. Find expressions for the following quantities, in terms of the dimensionless density parameter $(r_s/a_0)$, where $a_0$ is the Bohr radius, and with the units indicated:
   - the Fermi momentum $k_F$ ( in A ) , [Given: $1a_0 = 0.529$ A].
   - the Fermi energy $\epsilon_F$ (in eV ), [Given: 1 Rydberg = 13.6 eV ]
   - the Fermi temperature $T_F$ (in K). [Given: 1 eV = $k_B$ x 1.16 x $10^4 K$.]

   In each case, a detailed expression for the coefficient involved should be found so that if you had a calculator you would be able to evaluate the coefficient numerically.

3. Starting from the equation of state

$$PV = \frac{2}{3}\langle E \rangle$$

   where $\epsilon_F$ is the total internal energy of the gas, express the T = 0 value of the bulk modulus

$$B \equiv -V \left( \frac{\partial P}{\partial V} \right)_{T,N}$$

   which is the inverse of the isothermal compressibility $\kappa_T \equiv -\frac{1}{V}(\partial V/\partial P)_{T,N}$ V in terms of the Fermi energy $\epsilon_F$ and the density parameter $r_s$.

## Problem 5

According to the principles of quantum statistical mechanics, the pressure of black- body radiation inside a volume V may be calculated by treating the radiation as a photon gas, and using the relation

$$\overline{p} = \frac{1}{\beta}\frac{\partial \log Z}{\partial V}$$

where $\overline{p}$ is the mean pressure, Z the partition function, and $\beta = 1/k_B T$ is kept constant. You may assume that the volume V is a cubic box of edge length $L = V^{1/3}$, with walls maintained at temperature T.

1. Express the partition function Z in terms of the energies $\epsilon_s$, of a set of independent photon states (i.e., normal modes) in the volume, and use it to show that

$$\overline{p} = -\sum_s \frac{\partial \epsilon_s}{\partial V} \overline{n}_s$$

   where $n_s$ is the mean population of the state $s$ with energy $\epsilon_s$.

2. Using the above result, obtain an explicit relation between the mean pressure $\overline{p}$ and the mean energy density $\overline{u}(= \overline{E}/V)$ of the photon gas.

## Problem 6

Consider $N$ particles of a non-interacting spin-1 Bose gas of mass $m$. They are confined in three dimensions to a volume $V$ . Take $\epsilon = p^2/2m$

1. In the high-temperature, low-density limit, determine the partition function, the free energy, and the entropy.

2. In 1926, Einstein predicted that, at sufficiently low temperatures, a non-interacting Bose gas can undergo condensation in which the occupation number $N_0$ of the $p = 0$ state is macroscopic: i.e. $N_0/N$ is finite as $N \to \infty$. Taking the chemical potential to be zero, derive $N_0/N$ as a function $T$. Determine $T_E$, the Einstein condensation temperature, from the condition that $N_0(T_E) = 0$.

Any integrals that arise should be put in dimensionless form, but need not be evaluated.