Exercise Problems - Assignment 7

Statistical Mechanics and Thermodynamics

WS 2018

Due: 17th Dec 2018

JGU Mainz - Institut für Physik

Choose 4 of the 6 problems below. Each problem is worth 12 points.

Problem 1

For a ferromagnetic solid at low temperatures, the quantized waves of magnetization (spin waves) have their frequency ω related to their wave number k according to $\omega = Ak^2$ where A is a constant.

1. For a 3-dimensional solid, write down the density of states $D(\omega)$ for such an excitation.
2. Write an expression for the energy density for the spin waves.
3. Determine an expression for the heat capacity, and what is its temperature dependence at low temperatures?
4. At sufficiently low temperatures, which terms should give the largest contribution to the specific heat: phonons, electrons, or spin waves? Which would give the smallest contribution?

Problem 2

A gas is described by the following equations of state

$$P = \frac{U}{3V}, \quad U = bVT^4$$

where b is a constant and P, U, V, and T are the thermodynamic pressure, internal energy, volume and temperature.

1. Determine the entropy S as a function of U and V.
2. Determine the functional relationship between P and V along an adiabatic path.
3. Determine the functional relationship between P and V along an isothermal path.
4. Determine the work done by the system as it expands from an initial volume V_0 to a final volume $2V_0$ along an isothermal path with temperature T_0.
5. Determine the heat added to the system as it expands from an initial volume V_0 to a final volume $2V_0$ along an isothermal path with temperature T_0.

Problem 3

Consider a gas of non-interacting Bose particles with spin $S = 0$ and mass m. In the ultrarelativistic limit, one can approximate the dispersion relation by $E(p) = cp$.

1. Write down a general integral expression for the statistical average of the total number of particles not in the zero-energy ground state.
2. Determine the Bose-Einstein condensation temperature T_0 of the gas as a function of the gas density $\rho = N/V$.
3. Determine the fraction N_0/N of the particles in the zero-energy ground state as a function of temperature T and density ρ.

You may find the following formula useful:

$$\int_0^\infty \frac{z^{x-1}}{e^z - 1} dz = \Gamma(x)\xi(x)$$

x	3/ 2	5/ 2	3	5
Γ	$\sqrt{\pi}$ / 2	$3\sqrt{\pi}$ / 4	2	24
ξ	2.612	1.341	1.202	1.037

Problem 4

1. Show that the number of photons, N, in equilibrium at temperature T in a cavity of volume V is proportional to
$$V\left(\frac{k_B T}{\hbar c}\right)^3$$
2. Show that the heat capacity for this system is proportional to T^3.

Problem 5

1. Determine the chemical potential, at temperature T = 0 and at number density n for a non-interactmg, non-relativistic Fermi gas of spin-1/2 and mass $m(\epsilon = p^2/2m)$
2. Repeat the previous part for the relativistic case ($\epsilon = cp$)
3. Show that, at some critical density n, and T = 0, the proton-electron plasma starts a transition into the degenerate neutron gas. Neglect any interaction between the particles and consider the electron and proton systems as Fermi gases. Take into account the mass difference ΔM between the neutron and proton. Since $m_e \ll \Delta M$, the electrons must be treated relativistically. Assume that neutrons created in the course of this transformation leave the system. Neglect gravity.
4. Consider such a system in a box of volume V , with $n < n_c$ Determine the number of electrons Ne and their pressure as the volume is decreased somewhat below the volume where the transition occurs. Do not consider compression so high that complete conversion occurs.

Problem 6

According to the principles of quantum statistical mechanics, the pressure of black- body radiation inside a volume V may be calculated by treating the radiation as a photon gas, and using the relation

$$\overline{p} = \frac{1}{\beta}\frac{\partial \log Z}{\partial V}$$

where $\overline{p}$ is the mean pressure, Z the partition function, and $\beta = 1/k_B T$ is kept constant. You may assume that the volume V is a cubic box of edge length $L = V^{1/3}$, with walls maintained at temperature T.

1. Express the partition function Z in terms of the energies ϵ_s, of a set of independent photon states (i.e., normal modes) in the volume, and use it to show that
$$\overline{p} = -\sum_s \frac{\partial \epsilon_s}{\partial V}\overline{n}_s$$
where n_s is the mean population of the state s with energy ϵ_s.
2. Using the above result, obtain an explicit relation between the mean pressure $\overline{p}$ and the mean energy density $\overline{u}(= \overline{E}/V)$ of the photon gas.