Exercise Problems - Week 7

Statistical Mechanics and Thermodynamics

SoSe 2018

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Choose 4 of the 6 problems below. Each problem is worth 12 points.

Problem 1

1. Two containers of equal volume are filled with an equal number of moles of two different ideal monoatomic classical gasses. The containers are in thermal contact. The molecules of one of the gasses obey classical (Maxwellian, Boltzmann) statistics, while the molecules of the other gas obey Fermi-Dirac statistics. Which container has greater
   (a) pressure
   (b) internal energy
   (c) heat capacity at constant volume
   (d) entropy

2. Answer the same questions as for previous part, but comparing classical statistics and Bose-Einstein statistics. Do any of your answers depend on the temperature?

Problem 2

For a ferromagnetic solid at low temperatures, the quantized waves of magnetization (spin waves) have their frequency ω related to their wave number k according to $\omega = Ak^2$ where A is a constant.

1. For a 3-dimensional solid, write down the density of states $D(\omega)$ for such an excitation.

2. Write an expression for the energy density for the spin waves.

3. Determine an expression for the heat capacity, and what is its temperature dependence at low temperatures?

4. At sufficiently low temperatures, which terms should give the largest contribution to the specific heat: phonons, electrons, or spin waves? Which would give the smallest contribution?

Problem 3

The relation between the frequency ν and the wavelength λ for surface tension waves on the surface of a liquid of density ρ and surface tension σ is

$$\nu^2 = \frac{2\pi\sigma}{\rho\lambda^3}$$

Use a method analogous to the Debye theory of specific heats to find a formula, analogous to the Debye T^3 law, for the temperature dependence of the surface energy E of a liquid at low temperatures. The surface tension of liquid helium at $0K$ is $0.352 \times 10^{-3} N/m$ and its density is $0.145 g \cdot \text{cm}^{-3}$. From these data estimate the temperature range over which your formula for $E(T)$ is valid for liquid helium, assuming that each helium atom in the surface of the liquid possesses one degree of freedom. You may assume

$$\int_0^\infty \frac{x^{4/3}}{e^x - 1} dx = 1.68.$$

Bolzmann's constant $k_B = 1.38 \times 10^{-16}$ erg $\cdot K^{-1}$
Avogadro's number $N = 6.02 \times 10^{23}$ mole $^{-1}$
Planck's constant $\hbar = 1.05 \times 10^{-27}$ erg $\cdot$ sec

Problem 4

1. Show that the number of photons, N, in equilibrium at temperature T in a cavity of volume V is proportional to
$$V\left(\frac{k_B T}{\hbar c}\right)^3$$
2. Show that the heat capacity for this system is proportional to T^3.

Problem 5

According to the principles of quantum statistical mechanics, the pressure of black- body radiation inside a volume V may be calculated by treating the radiation as a photon gas, and using the relation

$$\overline{p} = \frac{1}{\beta}\frac{\partial \log Z}{\partial V}$$

where $\overline{p}$ is the mean pressure, Z the partition function, and $\beta = 1/k_B T$ is kept constant. You may assume that the volume V is a cubic box of edge length $L = V^{1/3}$, with walls maintained at temperature T.

1. Express the partition function Z in terms of the energies ϵ_s, of a set of independent photon states (i.e., normal modes) in the volume, and use it to show that
$$\overline{p} = -\sum_s \frac{\partial \epsilon_s}{\partial V}\overline{n}_s$$
where n_s is the mean population of the state s with energy ϵ_s.
2. Using the above result, obtain an explicit relation between the mean pressure $\overline{p}$ and the mean energy density $\overline{u}(=\overline{E}/V)$ of the photon gas.

Problem 6

Consider N particles of a non-interacting spin-1 Bose gas of mass m. They are confined in three dimensions to a volume V . Take $\epsilon = p^2/2m$

1. In the high-temperature, low-density limit, determine the partition function, the free energy, and the entropy.
2. In 1926, Einstein predicted that, at sufficiently low temperatures, a non-interacting Bose gas can undergo condensation in which the occupation number N_0 of the $p = 0$ state is macroscopic: i.e. N_0/N is finite as $N \to \infty$. Taking the chemical potential to be zero, derive N_0/N as a function T. Determine T_E, the Einstein condensation temperature, from the condition that $N_0(T_E) = 0$.

Any integrals that arise should be put in dimensionless form, but need not be evaluated.