Exercise Problems - Week 7

Statistical Mechanics and Thermodynamics

SoSe 2018

May 23, 2018

JGU Mainz - Institut für Physik

Choose 4 of the 6 problems below. Each problem is worth 12 points.

Problem 1

1. Find the Fermi energy at $T = 0$, ε_F, of a gas of N noninteracting spin one-half particles constrained to move in two dimensions within an area A.

2. The analogous of the pressure in two dimension is given by $-\partial\epsilon_n/\partial A$. Show that the 2-d pressure at $T = 0$ is given by $N\varepsilon_F/2A$.

Problem 2

A gas is described by the following equations of state

$$P = \frac{U}{3V}, \quad U = bVT^4$$

where b is a constant and P, U, V, and T are the thermodynamic pressure, internal energy, volume and temperature.

1. Determine the entropy S as a function of U and V.

2. Determine the functional relationship between P and V along an adiabatic path.

3. Determine the functional relationship between P and V along an isothermal path.

4. Determine the work done by the system as it expands from an initial volume V_0 to a final volume $2V_0$ along an isothermal path with temperature T_0.

5. Determine the heat added to the system as it expands from an initial volume V_0 to a final volume $2V_0$ along an isothermal path with temperature T_0.

Problem 3

Consider a gas of non-interacting Bose particles with spin $S = 0$ and mass m. In the ultrarelativistic limit, one can approximate the dispersion relation by $E(p) = cp$.

1. Write down a general integral expression for the statistical average of the total number of particles not in the zero-energy ground state.

2. Determine the Bose-Einstein condensation temperature T_0 of the gas as a function of the gas density $\rho = N/V$.

3. Determine the fraction N_0/N of the particles in the zero-energy ground state as a function of temperature T and density ρ.

You may find the following formula useful:

$$\int_0^\infty \frac{z^{x-1}}{e^z - 1} dz = \Gamma(x)\xi(x)$$

x	3/ 2	5/ 2	3	5
Γ	$\sqrt{\pi}$ / 2	$3\sqrt{\pi}$ / 4	2	24
ξ	2.612	1.341	1.202	1.037

Problem 4

The ground state density of a free-electron Fermi gas is conveniently parametrised by specifying the volume per conduction electron according to

$$\frac{4}{3}\pi r_s^3 \equiv \frac{V}{N}$$

1. Find the Fermi wavevector, k_F in terms of the electron density parameter rs.
2. Find expressions for the following quantities, in terms of the dimensionless density parameter (r_s/a_0), where a_0 is the Bohr radius, and with the units indicated:
   - the Fermi momentum k_F (in A) , [Given: $1a_0 = 0.529$ A].
   - the Fermi energy ϵ_F (in eV), [Given: 1 Rydberg = 13.6 eV]
   - the Fermi temperature T_F (in K). [Given: 1 eV = k_B x 1.16 x $10^4 K$.]

   In each case, a detailed expression for the coefficient involved should be found so that if you had a calculator you would be able to evaluate the coefficient numerically.
3. Starting from the equation of state

   $$PV = \frac{2}{3}\langle E\rangle$$

   where ϵ_F is the total internal energy of the gas, express the T = 0 value of the bulk modulus

   $$B \equiv -V\left(\frac{\partial P}{\partial V}\right)_{T,N}$$

   which is the inverse of the isothermal compressibility $\kappa_T \equiv -\frac{1}{V}(\partial V/\partial P)_{T,N}$ V in terms of the Fermi energy ϵ_F and the density parameter r_s.

Problem 5

1. Determine the chemical potential, at temperature T = 0 and at number density n for a non-interactmg, non-relativistic Fermi gas of spin-1/2 and mass $m(\epsilon = p^2/2m)$
2. Repeat the previous part for the relativistic case $(\epsilon = cp)$
3. Show that, at some critical density n, and T = 0, the proton-electron plasma starts a transition into the degenerate neutron gas. Neglect any interaction between the particles and consider the electron and proton systems as Fermi gases. Take into account the mass difference ΔM between the neutron and proton. Since $m_e \ll \Delta M$, the electrons must be treated relativistically. Assume that neutrons created in the course of this transformation leave the system. Neglect gravity.
4. Consider such a system in a box of volume V , with $n < n_c$ Determine the number of electrons Ne and their pressure as the volume is decreased somewhat below the volume where the transition occurs. Do not consider compression so high that complete conversion occurs.

Problem 6

According to the principles of quantum statistical mechanics, the pressure of black- body radiation inside a volume V may be calculated by treating the radiation as a photon gas, and using the relation

$$\overline{p} = \frac{1}{\beta}\frac{\partial \log Z}{\partial V}$$

where $\overline{p}$ is the mean pressure, Z the partition function, and $\beta = 1/k_B T$ is kept constant. You may assume that the volume V is a cubic box of edge length $L = V^{1/3}$, with walls maintained at temperature T.

1. Express the partition function Z in terms of the energies ϵ_s, of a set of independent photon states (i.e., normal modes) in the volume, and use it to show that

$$\overline{p} = -\sum_s \frac{\partial \epsilon_s}{\partial V} \overline{n}_s$$

   where n_s is the mean population of the state s with energy ϵ_s.

2. Using the above result, obtain an explicit relation between the mean pressure $\overline{p}$ and the mean energy density $\overline{u}(= \overline{E}/V)$ of the photon gas.