Exercise Problems - Week 4

Statistical Mechanics and Thermodynamics

SoSe 2018

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JGU Mainz - Institut für Physik

Choose 4 of the 6 problems below. Each problem is worth 12 points.

Problem 1

Consider an extremely relativistic gas consisting of weakly interacting N identical monoatomic molecules with energy momentum relationship $E = cp$, where c is the speed of light. The gas is confined to a volume V and is in thermal equilibrium at temperature T .

1. Calculate the partition function $Z_N(V, T)$ for the gas.
2. Calculate the Helmholtz free energy F.
3. Derive an equation of state of the gas.

Problem 2 Stat. Mech.

Imagine a set of N independent spins, with fixed positions and $J = 1$. In a uniform magnetic field, the spin energies are $E = \mu HS$, where $s = (1, 0, -1)$ labels the spin component along the magnetic field, H is the applied field, and μ is the magnetic moment. N is a very large number.

1. Find a general form for $n^+, n^0,$ and n^-, the expectation values for the numbers of spins with each of the three spin components, at a given temperature and field. Do not assume the high-temperature limit here.
2. What is $\langle E \rangle$, the total energy expectation value, in the high temperature limit (first temperature-dependent term)?
3. Find an expression for the entropy S, in the high-temperature limit. Here we are looking for two terms, the temperature-independent $T \approx \infty$ term and the first temperature-dependent term. (You may need Stirling's approximation, $\log N! \approx N \log N - N$, although one can do the problem without needing this formula.)
4. From the previous result, find the temperature dependence on the field H, as H is varied in an adiabatic process. For decreasing H, what is the process called?

Problem 3

The 1-dimensional Ising model is defined as a chain with "spins" σ_n, on each site n (n = 1, 2 . . . N) independently taking one of two values $\sigma_n = \pm 1$. The energy of this system can be written as follows:

$$H = -J \sum_{n=1}^{N-1} \sigma_n \sigma_{n+1}$$

where J is a positive constant.

1. Introduce " spins on a bond" $\tau_n = \sigma_n \sigma_{n+1}$ as new variables. Explain why the τ's are independent. Find the partition function.
2. Find the free energy and the heat capacity per site.
3. Find the asymptotic behavior of the heat capacity at $T \ll J$ and at $T \gg J$. Give a physical explanation for the dominant behavior at $T \ll J$.

Problem 4

A 2-dimensional classical ideal gas of N particles at a temperature T, is contained to an area A in the $x - y$ plane.

1. Find the partition function and the free energy for this system.
2. Determine the surface tension as a function of temperature and density (number of particles per unit area) $n = N/A$
3. If the same gas is placed in a field of constant force per particle, F, directed along the x-axis, determine the density n as a function of x.

Problem 5

Consider a classical ideal monoatomic gas at temperature T in a uniform gravitational field (i.e, an isothermal atmosphere). Assume that the gas atoms have mass, m, and that their distribution overall all possible heights from $z = 0$ (the surface of the earth) to $z = \infty$ is an equilibrium one. Denote the magnitude of the gravitational acceleration by g and assume that it can be taken to be a constant for all values of z.

1. Find the probability density governing the fraction of atoms per unit height at a height z.
2. Find the mean potential energy per atom.
3. Find the total internal energy and total heat capacity of the gas (i.e., the total amount of heat energy required to increase the temperature of the entire gas column by 1K).

Problem 6

A thermodynamic system consists of N spatially separated noninteracting subsystems. Each subsystem has non-degenerate energy levels 0, ϵ, 2ϵ, and 3ϵ. The system is in thermal equilibrium with a heat reservoir of absolute temperature $T = \epsilon/k_B$. Calculate the partition function, the mean energy, and the entropy of the thermodynamic system.